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PLATING UNDER LATERAL LOADS WITH SINGULAR POINTS ON THE COUNTER

Abstract. The article proposes special techniques for calculating plates with singular points on the counter that need no singular solution building. The analysis of circular plate with mixed support conditions is performed by means of overdetermined boundary collocation method using special Schwarz method. The analysis of the plate with three clamped and one free sides is performed through reducing two dimensional boundary value problems to those of one-dimensional type using special correcting function.

Keywords: plate, singular point, Schwarz method, correcting function, boundary collocation method

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ПОПЕРЕЧНЫЙ ИЗГИБ ПЛАСТИН С ОСОБЫМИ ТОЧКАМИ НА КОНТУРЕ

Аннотация. Пластины с особыми точками на контуре широко применяются в технике. Особыми точками являются входящие узлы пластин, места смены условий опирания на какой-либо стороне пластины, места сопряжения смежных сторон контура пластин и другие. При расчете пластин с особыми точками в окрестности последних решение имеет особенность, выделение которой связано с определенными трудностями. В статье предложены специальные приемы расчета пластин с особыми точками на контуре, не требующие построения сингулярной части решения. Расчет круглой пластины со смешанными условиями опирания выполнен методом переопределенной граничной коллокации с использованием специального приема Шварца, а расчет пластины с тремя защемленными и одной свободной от закреплений сторонами выполнен путем сведения двумерной краевой задачи к одномерной с использованием специальной корректирующей функции.

Ключевые слова: пластина, особая точка, прием Шварца, корректирующая функция, метод граничной коллокации

Introduction

Plates of different configurations and under various support conditions are widely used in mechanical engineering, aircraft engineering, shipbuilding and building industry.

Plates with singular points on the counter include circular and elliptical plates with their part of the counter being clamped and the other part being simply supported or free edged; rectangular plates with one of their two adjacent sides being clamped and the other being free; plates of complex shape in the plan with reentrant angles and other types of plates.

If plates of canonical geometry in the plan and at standard support conditions are easily calculated by means of well-known projection, meshless and finite element methods, plating analysis with singular points on the counter requires considerable efforts as singular solution is to be built at singular point boundaries.

Based on two boundary problem solutions — (1) circular plate under lateral load with a half of its counter being clamped and the second half being simply supported, and (2) square plate under lateral load, with its three sides being clamped and the forth side being free, — the article proves that one can obtain almost accurate results through techniques without building singular part of the solution.

1. Circular plate

Circular plate with the support conditions mentioned above is presented in dimensionless form in Fig. 1. It is assumed here that $r = \bar{r}/a$, $w = \bar{w}/h$, $p = \bar{p}a^4/Eh^4$, a, h are radius and thickness of a plate; $\bar{w} = (\bar{r}, \theta)$ is plate deflection; $\bar{p} = \text{const}$ is uniform load of intensities; $\bar{r}, \theta$ are polar coordinates; E is modulus of elasticity of isotropic material of plate; K are singular points.

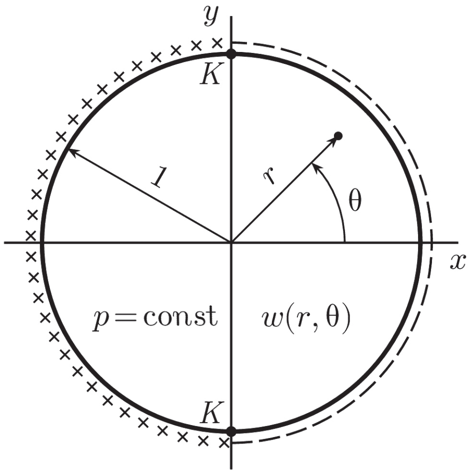

Fig. 1. Circular plate geometry

Under low deflexion $(\bar{w}_{\max} \le 0{,}2h)$ plate bending is described in dimensionless form by linear partial differential equation [1]:

$$\nabla^2\nabla^2 w = Cp, \qquad (1)$$

where operator $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$, $C = 12(1-\nu^2)$,

ν is the Poisson ratio.

Equation (1) should be integrated under boundary conditions with the consideration of symmetry above the x-axis:

$$w = 0,\ \frac{\partial w}{\partial r} = 0 \ \text{ when } \ r = 1,\ 0.5\pi < \theta \le \pi, \qquad (2)$$

$$w = 0,\ M_r = 0 \ \text{ when } \ r = 1,\ 0 \le \theta < 0.5\pi. \qquad (3)$$

Here $M_r = \bar{M}_r a^2 / Dh$,

$\bar{M}_r = -D\left[\dfrac{\partial^2 \bar{w}}{\partial \bar{r}^2} + \nu\left(\dfrac{1}{\bar{r}}\dfrac{\partial \bar{w}}{\partial \bar{r}} + \dfrac{1}{\bar{r}^2}\dfrac{\partial^2 \bar{w}}{\partial \theta^2}\right)\right]$ is radial bending moment; $D = Eh^3/12(1-\nu^2)$.

Solving boundary problem is performed by overdetermined boundary collocation method (OBCM) by means of Schwarz method [2] allowing to exclude from consideration singular points K.

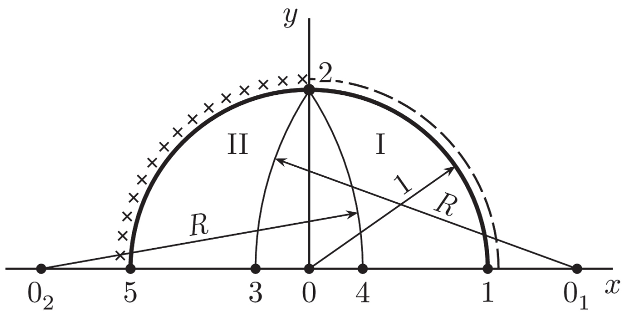

Fig. 2. Ananlysis model of the circular plate

We divide half of the plate located above the x-axis into two parts (I и II), angular points being numbered 1, 2, 3 and 4, 2, 5 (see Fig. 2). Lines 2–3 and 4–2 are arcs of circle radius R. Zone 3, 2, 4 is the area of the plates I and II overlapping.

Calculations on plates I and II are performed in a sequential order using iteration procedure.

The first step is to calculate the plate I under boundary conditions (3) on lines 1–2, in the absence of deflections (w) and normal bending moments (M_n) on lines 2–3 using OBCM.

Then we define deflections (w) and the angles of rotation of the normal (w'_n) on lines 4–2 and under these inhomogeneous and boundary conditions (2) we calculate the plate II using OBCM, determine w and M_n further calculate the plate I on lines 2–3. Iteration process is completed determining w, M_n, and w, w'_n on lines 2–3 and 4–2 to the specified degree of accuracy.

An outlined algorithm is implemented based on the deflection function by A. Clebsch [1]:

$$w = \sum_{t=1}^{s}\left[\left(A_t + r^2 A_{s+t}\right) r^{t_1}\cos(t_1\theta)\right] + cpr^4,\ \ c = \frac{3}{16}(1-\nu^2), \quad (4)$$

thus making the equation (1) become identical but not complying with boundary conditions (2) and (3).

To determine the constants, solution A_t, A_{s+t} bending function w'_n and bending moments M_n of the normal are presented as

$$w'_n = \sum_{t=1}^{s}\left(A_t\Psi_1 + A_{s+t}\Psi_2\right) + c4r^3\cos\beta, \qquad (5)$$

$$M_n = -\left[\sum_{t=1}^{s}\left(A_t\Psi_3 + A_{s+t}\Psi_4\right) + cf(r,\theta,\alpha)\right], \qquad (6)$$

where $\Psi_1 = t_1 r^{t_2}\cos\beta_2$; $\Psi_2 = r^t(t\cos\beta_2 + \cos\beta_0)$; $\Psi_3 = t_1 t_2 \nu_1 r^{t_3}\cos\beta_3$; $\Psi_4 = tr^{t_1}(t_1\nu_1\cos\beta_3 + 2\nu_2\cos\beta_1)$; $f(r,\theta,\alpha) = 4r^2[2\nu_2 + \nu_1\cos(2\beta)]$; $t_1 = t-1$; $t_2 = t-2$; $t_3 = t-3$; $\nu_1 = 1-\nu$; $\nu_2 = 1+\nu$; $\beta = \theta - \alpha$; $\beta_0 = t\theta - \alpha$; $\beta_1 = t_1\theta$; $\beta_2 = t_2\theta + \alpha$; $\beta_3 = t_3\theta + 2\alpha$, α is the angle between the normal of the corresponding plate sides I, II and the x-axis.

Here are some specific numerical results obtained when $p = 1$; $\nu = 0.3$; $R = 1.5$; $s = 80$ and with 220 points placed on sides 1—2 and plates 2—5, and 20 points of collocation located on sides 2—3 and 4—2.

Notice that first points of collocation are located in points 1, 3, 4, 5 of plates and last points are removed one step from the singular point 2 (*K*). Overdetermination degree (the system of linear equations) equals three $((220+20)/80)$.

Fig. 3 shows the results of the assigned task solving. If deflexions and radial bending moments on line 1—0-5 (along the *x*-axis) completely coincide with those obtained in [3], the values on the clamped part of the counter are significantly improved. In particular, the value $M_r = 9.57$ in the singular point from left is 3 times larger than that in [3]; in addition, boundary conditions over the entire counter of the plate and about the singular point K are met with high accuracy. Hence, $w_K^I = -3.43\cdot10^{-6}$, $M_{r,K}^I = -3.15\cdot10^{-5}$, $w_K^{II} = 3.03\cdot10^{-7}$, $\left(\dfrac{\partial w}{\partial r}\right)_K^{II} = -1.03\cdot10^{-5}$ are the largest on the counter of the plate and differ little from zero.

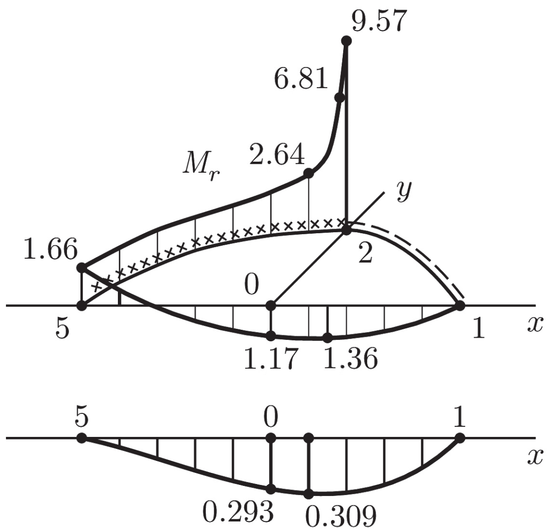

Fig. 3. Calculation results on the circular plate

Therefore, in the authors view, the results obtained are considered to be nearly accurate and building singular solution of a singular point according to the method by V. G. Karpunin [4] was not required.

2. Square plate

Square plate $(2a\times 2a)$ with three clamped sides and one free edged one is shown in dimensionless form in Fig. 4.

It is assumed that $x=\bar{x}/a$, $y=\bar{y}/a$, $w=\bar{w}/h$, $p=\bar{p}(2a)^4/Eh^4$, a, h are sizes in the plan and the plate thickness; $\bar{w}(\bar{x},\bar{y})$ are plate deflections; $\bar{x}, \bar{y}$ are rectangular coordinates; $\bar{p}=\text{const}$ is the uniform load of intensities; E is Young's modulus; K are singular points.

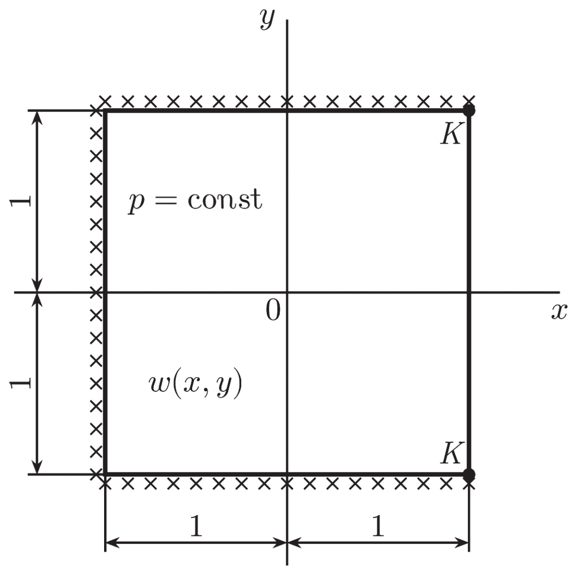

Fig. 4. Square plate geometry

Linear partial differential equation describing *plate bending normal to its plane* [1] in dimensionless form is given by

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \eta p, \tag{7}$$

where $\eta = 0.75(1-\nu^2)$, ν is the Poisson ratio.

In view of the *x*-axis symmetry, the equation (7) needs to be integrated under boundary conditions

$$x=-1,\ w=0,\ \frac{\partial w}{\partial x}=0, \tag{8}$$

$$x=1,\ M_x = \frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} = 0,\ V_x = \frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^3} = 0, \tag{9}$$

$$y=\pm1,\ w=0,\ \frac{\partial w}{\partial y}=0. \tag{10}$$

Here $M_x = \bar{M}_x a^2/Dh$, $V_x = \bar{V}_x a^3/Dh$, $D = Eh^3/12(1-\nu^2)$, $\bar{M}_x$, $\bar{V}_x$ are bending moments and generalized transverse force.

The paper [5] accurately solves the boundary problem assigned using finite element method (FDM) based on grid-based approximation of functionality that accounts for singular point presence and special two-step solution implementation approach.

There is the reason to believe that the results obtained by the authors are completely reliable, but it is recognized that the solution algorithm is highly tedious.

We cite the problem solution by means of the method more simple for understanding and implementation.

To exclude singular points *K* from consideration, we reduce two-dimensional boundary value problem to those of one-dimensional type.

The deflection function is shown as

$$w(x,y) = X(x)Y(y), \tag{11}$$

where $Y(y) = \Phi(y)\cdot L(y)$ is prescribed function; $X(x)$ is required function.

When $\Phi(y) = 1 - 2y^2 + y^4$, $L(y) = \exp\left(\sum_{k=1}^{3} k_i y^{2k}\right)$,

the function $Y(y)$ satisfies boundary conditions (10) and the correcting function $L(y)$ including three correcting coefficients k_i enhances the approximation of the bending function along the *y*-axis.

We insert (11) in the equation (7) and boundary conditions (8), (9) and perform the procedure of orthogonalizaiton.

As a result we get

$$i_1 X''''(x) + 2i_2 X''(x) + i_3 X(x) = i_4 \eta p, \tag{12}$$

$$X(x) = 0,\ X'(x) = 0 \tag{13}$$

when $x = -1$,

$$X''(x) + \nu_1 X(x) = 0,\ X'''(x) + \nu_2 X'(x) = 0 \tag{14}$$

when $x = 1$

where $i_1 = \int_0^1 Y(y)Y(y)dy$, $i_2 = \int_0^1 Y^{\bullet\bullet}(y)Y(y)dy$,

$i_3 = \int_0^1 Y^{\bullet\bullet\bullet\bullet}(y)Y(y)dy$, $i_4 = \int_0^1 Y(y)dy$; $\nu_1 = \nu i_2 / i_1$,

$\nu_2 = (2-\nu) i_2 / i_1$; $(')$, $(^{\bullet})$ is differentiation along *x* and *y* respectively.

We build the solution making the equation (12) identical but failing to comply with the boundary conditions (13) and (14).

We find consecutively [6]

$$\alpha = -i_2 / i_1,\ \beta = \sqrt{i_3 / i_1},\ \gamma = \sqrt{\beta^2 - \alpha^2},$$

$$\delta = \sqrt{\beta},\ \psi = \mathrm{arctg}(\gamma / \alpha),$$

$$n = \delta\cos(0{,}5\psi),\ m = \delta\sin(0{,}5\psi),\ F(x) = \exp(nx),$$

$$G(x) = \exp(-nx),$$

$$S(x) = \sin(mx),\ T(x) = \cos(mx),\ f(x) = F(x)\cdot T(x),$$

$$g(x) = F(x)S(x),$$

$$u(x) = G(x)T(x),\ v(x) = G(x)S(x),$$

then the desired solution may be written as

$$X(x) = C_1 f(x) + C_2 g(x) + C_3 u(x) + C_4 v(x) + \eta p i_4 / i_3, \tag{15}$$

where $C_k\ (k = 1\ldots 4)$ are the constants of integration obtained from the boundary conditions (13) and (14).

To include correcting coefficients k_i in the solution we employ the technique presented in [7] minimizing average residual under the load subject to constraints $\Delta_{avg} = 0.5(\Delta_x + \Delta_y) \le 0.075$, where Δ_x is standard deviation residual along the *x*-axis and $y = 0$, Δ_y is standard deviation residual along the *y*-axis when $x = 1$.

It is easily found within the accuracy of ± 0.0001 that when $k_1 = 0.0014$, $k_2 = -0.0063$, $k_3 = -0.0025$, residuals Δ_x and Δ_y equal 0.06786 and 0.08204, and the average residual is $\Delta_{avg} = 0.07495$ that is lower than 0.075.

Fig. 5 shows some of the solution results when $p = 1$, $\nu = 1/6$. The values of non-dimensional deflexions and bending moments obtained by FDM in [5] on the grid 32×32 are given within parentheses.

As can be seen, the consistency between results is wholly satisfactory but the algorithm implemented in the present paper is easier and less tedious than that implemented in [5].

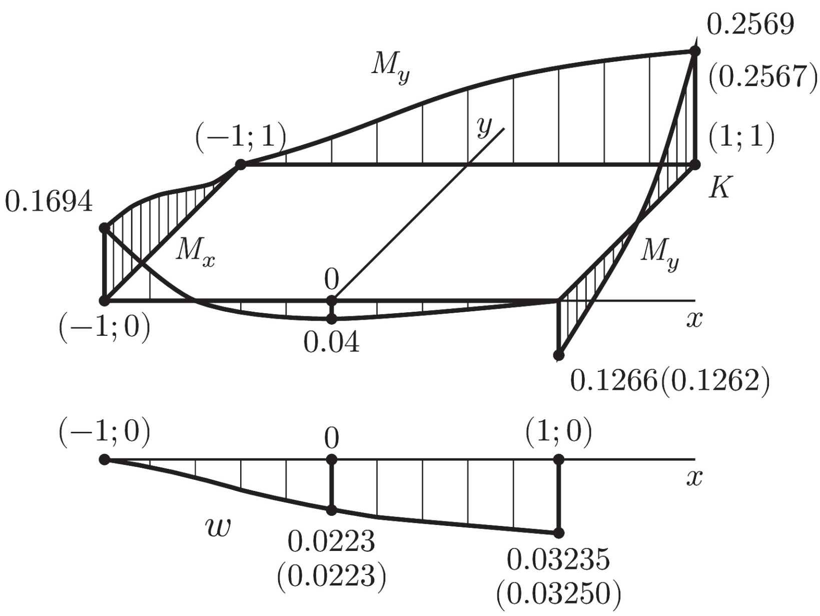

Fig. 5. Calculation results on the square plate

Conclusions

To conclude, boundary value problems of bending plates (and slabs) with singular points to be solved using techniques presented in the article are extensive. In addition, reliable determination of deflexions and stresses arising in plates ensures their successful practical application as bearing structures.

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